we prove that atleast for gradient Ricci solitons completeness of the underlying metric $g \Leftrightarrow$ completeness of the v.f. ∇f.

This result is originally due to Zhang '09.

Theorem (Z. Zhang 2009)

Suppose (M^n, g, f, λ) is a gradient RS w/ (M^n, g) complete. Then ∇f is a complete v.f.

Proof:- Recall that we have

$$R + |\nabla f|^2 - \lambda f = C$$

and we just proved that $R \geq 0$ if $\lambda \geq 0$ and $R \geq \frac{\lambda n}{2}$ if $\lambda < 0$.

$$\Rightarrow \quad |\nabla f|^2 = \lambda f + C - R$$

$$\leq \lambda f + C \text{ w/ this } C = C(\lambda, n) \geq 0.$$

for the case $\lambda \neq 0$, $\Rightarrow$ $h = \lambda f + C$ satisfies $h \geq 0$

and $|\nabla h|^2 = |\lambda \nabla f|^2 \leq |\lambda|^2 h$

$$\Rightarrow \quad |\nabla \sqrt{h}| \leq \frac{|\lambda|}{2}.$$

Let $q \in M^n$ and integrate along any minimizing unit speed geodesic $\gamma : [0, r(q)] \to M^n$, we get

$$\sqrt{h}(q) - \sqrt{h}(p) = \int_0^{r(q)} \langle \nabla \sqrt{h}(\gamma(s)), \gamma'(s) \rangle ds$$

$$\leq \int_0^{r(q)} |\nabla \sqrt{h}| ds \leq \frac{|\lambda|}{2} r(q)$$

$$\Rightarrow \sqrt{h}(q) \leq \frac{|\lambda|}{2} r(q) + \sqrt{h}(p)$$

$$\Rightarrow |\nabla f|(q) \leq |\lambda| r(q) + C' \quad \text{--- } (1)$$

and similar thing holds for $\lambda = 0$.

So overall what we get in eq. (1) is that the v.f. ∇f grows at most linearly in distance.

We have the following general result.

Lemma:- Let $X \in \Gamma(TM)$. If g is a complete metric on M^n s.t. $|X|_g(q) \leq C(d(p,q)+1)$ for some constant C and a fixed point $p \in M$ then X is a complete v.f.

Proof:- let $q \in M^n$ be arbitrary and let $\gamma : (a,b) \to M^n$ $-\infty \leq a < 0 < b \leq \infty$ be any maximal integral curve of X w/ $\gamma(0) = q$.

We know that $\exists$ a constant $C \geq 0$ s.t. for any $t \in [0,b)$ we have

$$r(\gamma(t)) \leq r(q) + d(q, \gamma(t))$$

$$\leq r(q) + \int_0^t |X|(\gamma(\tau))\, d\tau$$

$$\leq r(q) + Ct + C\int_0^t r(\gamma(\tau))\, d\tau$$

Gronwall's inequality: If $u(t)$ satisfies

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s)\, ds \quad \text{then}$$

w/ $\beta(t) \geq 0$ and α is non-decreasing then

$$u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)\,ds\right).$$

for us $u(t) = \pi(\gamma(t))$, $\beta(t) = C$

$$\alpha(t) = \pi(q) + Ct$$

$$\Rightarrow \quad \pi(\gamma(t)) \leq (\pi(q) + Ct)\,e^{Ct} \quad \forall\, t < b.$$

$\therefore \lim_{t \to b} \pi(\gamma(t)) = \infty$ only if $b = \infty$.

Similarly for $t \to a$, we must have $a = -\infty$.

$\Rightarrow X$ is complete.

<u>Theorem</u> (Ivey) Any closed steady or expanding soliton is Einstein. Any shrinking soliton has $R > 0$.

<u>Proof</u>:- (M^n, g, X, λ) is a compact Ricci soliton w/ $\lambda \leq 0$.

i.e., $\quad Ric + \frac{1}{2}\mathcal{L}_X g = \frac{\lambda}{2} g$

$$\Rightarrow \quad R + \operatorname{div} X = n\lambda/2$$

which on integrating $\Rightarrow$ $\Rightarrow r = R_{avg} = \dfrac{\int R \, d\mu}{\mathrm{Vol}(M)} = \dfrac{n\lambda}{2} \le 0.$

$$\int R \, d\mu = \frac{n\lambda}{2} \mathrm{vol}(M) \le 0. \quad \text{—} (1)$$

Moreover, divergence of the RS eq $\Rightarrow$

$$\nabla^i \left(R_{ij} + \frac{1}{2} \nabla_i X_j + \frac{1}{2} \nabla_j X_i \right) = \frac{\lambda}{2} \nabla^i g_{ij}$$

$$\Rightarrow \frac{1}{2} \cancel{\nabla_j R} + \frac{1}{2} \Delta X_j + \frac{1}{2} \cancel{\nabla_j \mathrm{div} X} + \frac{1}{2} R_{jm} X^m$$

$$= 0$$

$$\Rightarrow \Delta X_j + R_{jm} X^m = 0.$$

moreover, $\because$

$\Delta_X R = \Delta R - \langle X, \nabla R \rangle$ and we want to relate $\Delta_X R$ to the Ricci tensor.

We start w/

$$\nabla^i \nabla_i X_j + R_{jm} X^m = 0$$

$$\Rightarrow \nabla^j \nabla^i \nabla_i X_j + \frac{1}{2} \langle \nabla R, X \rangle + R_{jm} \nabla^j X^m = 0$$

$$\Rightarrow \nabla^i \nabla_j (\nabla_i X_j) - R^{ji}{}_{im} \nabla^m X_j - R^{ji}{}_{jm} \nabla_i X^m$$

$$+ \frac{1}{2} \langle \nabla R, X \rangle + R_{jm} \nabla^j X^m \qquad = 0$$

$$\Rightarrow \nabla^i \nabla_j (\nabla_i X^j) + R^{im} \frac{1}{2} (\nabla_i X_m + \nabla_m X_i)$$

$$+ \frac{1}{2} \langle \nabla R, X \rangle = 0$$

$$\Rightarrow \nabla^i \left[\nabla_i \nabla_j X^j + R_{im} X^m \right] + R^{im} \left[-R_{im} + \frac{\lambda}{2} g_{im} \right]$$

$$+ \frac{1}{2} \langle \nabla R, X \rangle = 0$$

$$\Rightarrow \Delta (\operatorname{div} X) + \frac{1}{2} \langle \nabla R, X \rangle + R_{im} \nabla^i X^m - |Ric|^2$$

$$+ \frac{1}{2} \langle \nabla R, X \rangle \qquad + \frac{\lambda}{2} R \qquad = 0$$

$$\Rightarrow -\Delta R + \langle \nabla R, X \rangle - |Ric|^2 + \frac{\lambda}{2} R + R_{im} \nabla^i X^m = 0$$

$$\Rightarrow \quad \Delta_X R - \lambda R + 2 |Ric|^2 = 0 .$$

$\therefore\ r = \int R\, d\mu = +\frac{n\lambda}{2} vol(M) = \text{constant}$

and recall that $r = \frac{n\lambda}{2}$

$$\Rightarrow \Delta_x (R-r) + 2\underbrace{\left|Ric - \frac{r}{n}g\right|^2}_{2|Ric|^2 - \frac{4Rr}{n} + \frac{2r^2}{n}} + \frac{2r}{n}(R-r) = 0 \qquad \text{--- } (2)$$

$\therefore\ M^n$ is compact $\Rightarrow$ R achieves its minima say R_{min} at some $x_0 \in M^n$ and $\therefore$ at that point.

$$2\underbrace{\left|Ric - \frac{r}{n}g\right|^2}_{\geq 0} + \underbrace{\frac{2r}{n}(R-r)}_{\geq 0 \text{ as } r = \frac{\int R\, vol}{vol(M)} \geq R_{min}} \leq 0$$

$\therefore\ Ric = \frac{r}{n}g$ and $R_{min} = r \Rightarrow R = r.$

If the soliton is shrinking $\Rightarrow$ $\lambda < 0$

so again from the strong max. principle

we get $R \geq 0$. Also either $R \equiv 0$ or $R > 0$.

If $R \equiv 0$ we get $Rc \equiv 0$

$\Rightarrow \quad \mathcal{L}_X g = \lambda g \Rightarrow \operatorname{div} X = n\lambda$

which would force $\lambda = 0$. which is not possible.

$\boxed{}$

* general Lemma :-

Consider the ODE

$$\frac{dx}{dt} = g(x(t)).$$

Lemma :- If $x(t)$, $a \le t \le b$ is a maximal integral curve then either

i) $b = +\infty$

ii) or $\exists$ a sequence of times $t_i \in [a, b)$ s.t. $t_i \to b$ and $|x(t_i)| \to \infty$.

Proof :- We prove that if not ii) then i). Suppose $|x(t)| < \infty$ $\forall\, t < b$. Let